THE GENERAL FIELD THEORY OF SCHOUTEN AND VAN DANTZIG

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PREFACE

The problem of unified field-theories is to find a geometry, which will represent both the gravitational and electromagnetic phenomena just as the Riemannian geometry of general relativity represents gravitational phenomena alone. The recent attempts depend on the use of a projective geometry employing five homogeneous coordinates.

In this summary of a course of three lectures, delivered at the Lucknow University, my aim has been to give an account of the field-theory of Schouten and Van Dantzig and of some fresh contributions to that theory.

In the first lecture is given a short account of the results in projective relativity leading up to the theory of Schouten and Van Dantzig.

In the second lecture field-equations in the Schouten and Van Dantzig theory are given and the identities between these field-equations have been derived.

In the third lecture the truth of these identities is verified by direct substitutions and the connexion is shown between the general relativity theory and the present theory. An attempt has also been made to introduce the current vector and Λ, the universal constant proportional to the curvature of the world, into the field equations and the identities.

A bibliography of original sources cited in the text is appended at the end. In the body of the book they are referred to by their serial numbers.

It is a great pleasure to record here my indebtedness to Professor B. Sahni, F.R.S. for asking me to deliver these lectures and making their publication possible.

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First Lecture

Results in Projective Relativity

1. *Introduction*—The problem of unified field theory is to find a geometry, which will represent both gravitational and electro-magnetic phenomena just as Riemannian geometry of general relativity represents gravitational phenomena alone. It was particularly unsatisfactory that the electro-magnetic field and the gravitational field should enter the theory as two fundamental mutually independent concepts. After years of effort some logical unification has been achieved. The most recent attempts depend on the use of a projective geometry employing five homogeneous coordinates. This is a step in the geometrization of physics. It may be said to have started with Faraday's lines and tubes of force and electricity filling all space. Then we have the attempts of Einstein, who from 1915 to 1933 has been occupied in finding a geometry to

study physics. Weyl's (5) attempts in 1917-18 and those of Eddington (7) in 1921 to bring macroscopic physics into geometrical form may be mentioned here. The general field-theory is the direct descendant of the five-dimensional theory of Kaluza (8) and Klein (10). This theory has received a clear and detailed investigation at the hands of Schouten and van Dantzig (20,21,23,33) and in this lecture we give a short account of the results leading to the theory before we take up the actual field-equations and some fresh contribution to the theory in the following lectures. As will be shown, the theories of Veblen and Hoffmann (22) and Einstein and Mayer (16) are particular cases of this theory. In the Riemannian geometry of ordinary general relativity a unification of electro-magnetic and gravitational phenomena is impossible. All unification theories make use in some way of a fifth coordinate in the local spaces. Now it seems impossible to give this fifth coordinate any physical meaning and this leads to the conception due to Veblen that local spaces are projective spaces and the five coordinates are the well known homogeneous coordinates of projective geometry.

Using projective local spaces, pseudo-parallelism can be generalised into a mapping of local

spaces on each other in a projective way. Cartan created for the first time projective geometries of a very general kind based on this principle. But these geometries are too general for relativity, since there has to be some kind of a metric. To get such a metric Veblen introduced in each local space a non-degenerate quadric and imposed the condition that this quadric is an invariant of the process of mapping local spaces on each other. If we take this quadric as the unit sphere, there is such analogy with Riemannian geometry, but there is a big difference, the contact point and the hyperplane at infinity are no longer invariants of the mapping process.

We can take the quadric as unit-sphere. Then the difference with the Riemannian geometry is that the contact-point and the hyperplane at infinity are no longer invariant.

2. *Homogeneous coordinates*—We describe the four-dimensional space-time—world by five homogeneous coordinates,

$$(x^0, x^1, x^2, x^3, x^4) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2\cdot 1)$$

These five numbers define a point so that we have ∞^5 points. Two points x and y are said to be coincident if

$$\frac{y_0}{x_0} = \frac{y^1}{x^1} = \frac{y^2}{x^2} = \frac{y^3}{x^3} = \frac{y^4}{x^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.2)$$

A set of coincident points defines a *spot*. A *spot* is physically a point-event in space-time.

Subject the coordinates to a transformation

$$x^{\nu\,\prime} = x^{\nu\,\prime}(x^0, x^1, x^2, x^3, x^4), \; (\nu = 0, 1, 2, 3, 4) \quad (2\cdot3)$$

where these new coordinates are homogeneous functions of degree one, not necessarily linear, of the old coordinates. A group of these transformations is called H_5.

We also consider transformations of points,

$$x^{\nu} = \rho x^{\nu} \quad (\nu = 0, 1, 2, 3, 4) \quad \ldots\ldots \quad (2\cdot4)$$

where ρ is an arbitrary function of x^{ν}, homogeneous and of degree zero. These transformations leave invariant every spot. We denote the group of these transformations by F. These are peculiar to the projective theory of relativity. It gives change from point to point.

Tensors of Einstein's general Relativity are now called *affinors* because they belong to the affine geometry. We now introduce the analogues of tensors, which we call "*projectors*." A function of x^0, x^1, x^2, x^3, x^4, which is invariant under the transformation of the group H_5 and which acquires a factor ρ^{ν} under transformation F is called a *scalar* of degree r.

A set of five functions, which under H_5 have the law

$$V^{\nu'} = \sum_{\nu} \frac{\partial x^{\nu'}}{\partial x^{\nu}} V^{\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\cdot5)$$

and under F acquire the factor ρ^{ν} so that

$$\overline{V}^{\nu} = \rho^{\nu} V^{\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\cdot6)$$

constitutes a contravariant projector of rank one and degree r or a contravariant point of degree r. This corresponds to a contravariant vector of the ordinary tensor-calculus.

A set of five functions, which under H_5 have the law of transformation

$$W_{\nu'} = \sum_{\nu} \frac{\partial x^{\nu}}{\partial x^{\nu'}} W_{\nu} \quad \ldots\ldots\ldots\ldots\ldots \quad (2.7)$$

and under F acquire ρ^{ν} is called a covariant point of degree r. Schouten and van Dantzig call it a covariant projector of valence one and degree r.

Similarly a projector of contravariant valence p and covaraint valence q and degree r is a set of 5^{p+q} homogeneous functions of the coordinates of degree r, which transform under H_5 like a product of p contravariant and q covariant points and under F acquire ρ^r.

Let ζ^1, ζ^2, ζ^3, ζ^4 be the ordinary non-homogeneous coordinates in four-dimensional space-time.

The differentials $d\zeta^1$, $d\zeta^2$, $d\zeta^3$, $d\zeta^4$ are regarded as coordinates in another 4-dimensional space called the *tangent-space* at the point. In this local tangent-space we introduce a Euclidean (Minkowski) metric $ds^2 = d\tau^2 - (dx^2 + dy^2 + dz^2)$. The theory of these tangent-spaces together with the underlying space constitutes a Riemannian metric. The Minkowski-coordinates at any point (T, X, Y, Z) are functions of the general coordinates ζ^1, ζ^2, ζ^3, ζ^4 in terms of which we describe the whole space so that $ds^2 = \sum\limits_{pq} g_{pq}\, d\zeta^p\, d\zeta^q$.

Similarly in the new theory at each spot of space-time we have a local tangent-space $E_4{}^*$, which is a *projective space*. To begin with we define $E_4{}^*$ attached to any spot P as a space having coordinates z°, z^1, z^2, z^3, z^4, which undergo the transformation

$$z^{\nu'} = \sum_{\nu} \frac{\partial x^{\nu'}}{\partial x^{\nu}} z^{\nu} \qquad \text{.......................} \qquad (2\cdot 8)$$

when the coordinates of the underlying space are subjected to transformation H_5. The contact points of degree zero, which exist at this spot P are in one-to-one correspondence with the points of the $E_4{}^*$ and may, therefore, be identified with them.

By Euler's theorem on Homogeneous functions

$$x^{\nu'} = \sum \frac{\partial x^{\nu'}}{\partial x^{\nu}} x^{\nu} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2\cdot 9)$$

Hence, $(x^{\circ}, x^1, x^2, x^3, x^4)$ are not only homogeneous coordinates of a point of space-time but also coordinates of a point of degree one of the E_4*. The corresponding point of the E_4* identified with the point x^{ν} of the space-time is called the point of contact of the E_4*.

3. *The fundamental quadric*—In every local space E_4* let us introduce a quadric (3-dimensional quadratic hypersphere φ) which does *not* contain the point of contact x^{ν},

$$\sum_{\mu\,\nu} G_{\mu\,\nu}\; V^{\mu} . V^{\nu} = 0; \quad \mathrm{Det}\,(G_{\mu\,\nu}) \neq 0; \;\ldots\ldots\; (3\cdot 1)$$

$G_{\mu\,\nu}$ is a projector of covariant valence 2. So we normalize $G_{\mu\,\nu}$ by the condition that

$$\sum G_{\mu\,\nu}\; x^{\mu} . x^{\nu} = -\omega^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3\cdot 2)$$

where ω is a positive constant of dimensions of length. $\omega = \dfrac{kh}{2nc}$, k being constant of gravitation, c is the velocity of light and $\mu\nu = 0, \; 1, 2, 3, 4$.

In Einstein's theory $\sum_{ij} g_{ij}\, dx^i\, dx^j$ $(ij = 1, 2, 3, 4)$ gives ten g's determining the gravitational-field. Here we have fifteen G's of which fourteen are at our disposal since one is normalized. This supplies ten g_{ij} to account for the gravitational-field and four for electro-magnetic properties. As in Riemannian geometry using the $G_{\mu\nu}$'s to raise and lower the indices we get,

$$V_\lambda = \sum_\mu G_{\lambda\mu} V^\mu \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3\cdot 3)$$

The point of contact x^λ will have a polar hyperplane with respect to the fundamental quadric

$$\left[\sum_{\lambda\mu} G_{\lambda\mu} x^\lambda V^\mu = 0\right]$$

This is specified by the covariant point

$$\left.\begin{array}{ll} & x_\lambda = \sum_\mu G_{\lambda\mu}\, x^\mu \\ \text{so} & \sum_\lambda x_\lambda x^\lambda = -\omega^2 \\ \text{or} & \sum_\lambda q_\lambda \cdot q^\lambda = -1 \\ \text{writing} & x^\lambda = \omega q^\lambda \text{ and } x_\lambda = \omega q_\lambda \end{array}\right\} \quad (3\cdot 4)$$

As in elementary projective geometry, we can find a Euclidean metric in which the fundamental quadric φ becomes a hypersphere, the point of contact its centre, the polar hyperplane of the point of contact becoming the plane at infinity.

Since the hyperplane at infinity is the plane $\sum\limits_{\lambda} q_\lambda V^\lambda = 0$, in this metric the square of the element of the length is $dx^2 + dy^2 + \ldots\ldots\ldots\ldots$

where
$$x = \omega . \frac{\sum\limits_{\lambda} a_\lambda V^\lambda}{\sum q_\lambda V^\lambda}$$
$$y = \omega \frac{\sum b^\lambda \ V^\lambda}{\sum q_\lambda \ . \ V^\lambda}, \text{ and so on.}$$

The equation of the quadric is
$$(\sum_\lambda a_\lambda V^\lambda)^2 + (\sum_\lambda b_\lambda \ V^\lambda)^2 + \ldots\ldots\ldots (\sum_\lambda q_\lambda V^\lambda)^2 = 0$$
and we assume that the point of contact which is the centre of the hypersphere has its $x, y, z, \ldots\ldots$ all zero.

Hence $$\sum a_\lambda x^\lambda = 0, \sum b_\lambda x^\lambda = 0, \ldots$$

This equation must be the same as
$$\sum_{\lambda \mu} G_{\lambda \mu} V^\lambda . V^\mu = 0.$$

Hence we must have
$$K\, G_{\lambda \mu} = a_\lambda . a_\mu + b_\lambda . b_\mu + \ldots\ldots\ldots q_\lambda q_\mu .$$

Since $$\sum_{\lambda \mu} G_{\lambda \mu} x^\lambda . x^\mu = -\omega^2 \text{ by } \qquad (3.4)$$
$$K \sum_{\lambda \mu} G_{\lambda \mu} x^\lambda . x^\mu = \sum_{\lambda \mu} a_\lambda . a_\mu x^\lambda . x^\mu + \ldots\ldots\ldots$$
$$\sum_{\lambda \mu} q_\lambda \ q_\mu x^\lambda . x^\mu \text{ or } - K \omega^2 = -\omega^2$$
and therefore $K = 1$ and $G_{\lambda \mu} = a_\lambda \ a_\mu + \ldots\ldots q_\lambda . q_\mu .$

Denoting by l the distance in this metric from the point of contact to an arbitrary point V^λ in the E^*_4, we get

$$l^2 = \frac{\omega^2}{V^2}\sum_{\lambda\mu}(G_{\lambda\mu} + q_\lambda\, q_\mu)V^\lambda \,.\, V^\mu \ldots\ldots \qquad (3.5)$$

where $V = -\sum_\lambda q_\lambda V^\lambda$ is called the *weight* of the contravariant point V^λ.

Excess—Suppose with some projector we perform on the points the transformation $F, \bar{x}^\nu = \rho\, x^\nu$ where ρ is a constant and then perform the transformation $x^{\nu\prime} = \rho^{-1}\, x^\nu$ of H_5. So the new coordinates of the new points are the same as the old coordinates of the old points. If the projector acquires the factor $\rho\varepsilon$, then ε is called the *excess* of the projector. Suppose the projector is of degree r in contravariant valence t, and covariant valence s. When we perform $\bar{x}^\nu = \rho x^\nu$ of F, the projector acquires ρ^ν; when we perform $x^{\nu\prime} = \rho^{-1}\, x^\nu$ of H_5 the projector acquires ρ^{s-t} $\therefore$ $\rho^\omega = \rho^r\, \rho^{s-t}$ and hence

$$\varepsilon = r + s - t.$$

Unless the contrary is stated all projectors henceforward will be supposed to be of excess zero. The point of contact x^ν has $r = 1$, $s = 0$, $t = 1$

and so $\varepsilon = 0$. $G_{\mu\nu}$ has $r = -2$, $s = 2$, $t = 0$ and hence $\varepsilon = 0$.

The derivative of a scalar is a vector only if the scalar is of excess zero.

Now introduce the description of the space-time world by four non-homogeneous coordinates ζ^1, ζ^2, ζ^3, ζ^4. These are homogeneous functions of zero degree of x^0, x^1, x^2, x^3, x^4. The space of $(\zeta^1, \zeta^2, \zeta^3, \zeta^4)$ (ordinary space-time) is denoted by X_4. In this space as in Einstein's general relativity we introduce at each point a tangent space E_4. The coordinates in it are $d\zeta^1$, $d\zeta^2$, $d\zeta^3$, $d\zeta^4$. We identify the points $(\zeta^1, \zeta^2, \zeta^3, \zeta^4)$ of the X_4 with the spots of the H_4. We write

$$A^k_\nu \equiv \frac{\partial \xi^k}{\partial x^\nu}, \quad \left.\begin{array}{l} k = 1, 2, 3, 4 \\ \nu = 0, 1, 2, 3, 4 \end{array}\right\} \quad \ldots \quad (3.6)$$

This is called an affinor-projector since

$$A^{k'}_{\nu'} = \sum_{k\mu} \frac{\partial \xi^{k'}}{\partial \zeta^k} \cdot \frac{\partial x^\mu}{\partial x^\mu} \cdot A^k_\mu; \quad A^{k'}_{\nu'} = \frac{\partial \xi^{k'}}{\partial x^\nu}$$

$$= \sum \frac{\partial \xi^{k'}}{\partial \xi^k} \cdot \frac{\partial \xi^k}{\partial x^\mu} \cdot \frac{\partial x^\mu}{\partial x^{\nu'}}.$$

To every covariant vector W_k, A^k_μ sets up a correspondence with a covariant point by the equation

$$W_\mu = \sum_k A^k_\mu W_k.$$

By Euler's theorem on homogeneous functions

$$\sum_{\mu} x^{\mu} \cdot \frac{\partial \zeta^k}{\partial x^{\mu}} = \sum_{\mu} x^{\mu} \; A^k_{\mu} = 0 \text{ or } \sum_{\mu k} q^{\mu} \; A^k_{\mu} W_k = 0 \quad \ldots\ldots\ldots \quad (3\cdot7)$$

since ζ's are homogeneous functions of degree zero of x'. To the affinor projector A^k_{μ} let us associate a dually corresponding A^{ν}_k defined by

$$\left.\begin{aligned} \sum_{\nu} A^{\nu}_j \; A^k_{\nu} &= \delta^k_j \\ \sum_{\nu} q_{\nu} \; A^{\nu}_j &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (3\cdot8)$$

There exists a one-to-one correspondence between covariant and contravariant points of weight and excess zero of $E_4{}^*$ and covariant and contravariant vectors of E_4 respectively given by

$$V^k = \sum_{\mu} A^k_{\mu} \; V^{\mu} \text{ and } V^{\nu} = \sum_{k} A^{\nu}_k \; V^k$$

The Projector A^{ν}_{μ} :—

Define A^{ν}_{μ} as

$$A^{\nu}_{\mu} = \sum_{k} A^k_{\mu} \; A^{\nu}_k \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.9)$$

From the equations

$$\left.\begin{aligned} \sum_{\nu} A^{\nu}_j A^k_{\nu} &= \delta^k_j \\ \sum_{k} q_{\nu} A^{\nu}_j &= 0 \\ \text{and} \quad \sum_{k} A^k_{\mu} A^{\nu}_k &= A^{\nu}_{\mu} \end{aligned}\right\}$$

we get in general

$$A^{\nu}_{\mu} = \delta^{\nu}_{\mu} + q^{\nu} q_{\mu} \quad \ldots\ldots\ldots\ldots\ldots \quad (3.9)$$

Hence

$$\sum_{\mu} q^{\mu} A^{\nu}_{\mu} = \sum_{\mu} q^{\mu} (\delta^{\nu}_{\mu} + q^{\nu} q_{\mu}) = q^{\nu} - q^{\nu} = 0.$$

Similarly $$\sum_{\nu} q_{\nu} A^{\nu}_{\mu} = 0.$$

Hence we get that if V^{ν} is a contravariant point of weight zero (and so corresponds to a vector) then

$$\sum_{\nu} A^{\mu}_{\nu} V^{\nu} = \sum_{\nu} (\delta^{\mu}_{\nu} + q^{\mu} q_{\nu}) V^{\nu} = V^{\mu}$$

Similarly $$\sum_{\mu} A^{\mu}_{\nu} W_{\mu} = W_{\nu},$$

if W_{ν} is a covariant point of weight zero. Thus contravariant and covariant points of weight zero (contravariant and covariant vectors) are unaltered by transvection with A^{μ}_{ν}.

4. *Correspondence between E_4 and E^*_4*:—Define the difference of two spots λV^{ν} and μV^{ν} by

$$\frac{V^{\nu}}{V} - \frac{U^{\nu}}{U} \qquad \dots\dots\dots\dots\dots\dots \quad (4.1)$$

where V and U are their weights.

This is a contravariant point of weight zero. as is seen by

$$\sum \frac{q_{\nu} V^{\nu}}{V} - \sum \frac{q_{\nu} V^{\nu}}{V} = 1 - 1 = 0$$

and so corresponds to a vector. In particular the difference of a spot V^ν and the spot of the point of contact is

$$\frac{V^\nu}{V} - q^\nu$$

since $$q_\nu\, q^\nu = -1$$

So the difference of the spot of V^ν and the spot of the point of contact corresponds to a vector r^k where

$$r^k = \sum_\nu A^k_\nu \left(\frac{1}{V} V^\nu - q^\nu\right) = \frac{1}{V} \sum_\nu A^k_\nu V^\nu$$

or conversely $$\frac{1}{V} V^\nu - q^\nu = \sum A^\nu_k r^k \qquad \cdots (4\cdot3)$$

or $$V^\nu = V\left(\sum_k A^\nu_k r^k + q^\nu\right)$$

These equations set up a one-to-one correspondence between the points r^k of the local E_4 (r^k being a vector from the origin of the E_4 to the points in E_4) and those spots V^ν of the E^*_4, which do not lie in the hyperplane q_λ. The point of contact of the $E_4{}^*$ corresponds to the origin of E_4 and the straight lines through these points correspond. The spots in E^*_4, which lie in the hyperplane q_λ correspond to the points at infinity. Thus E^*_4 is identified with E_4.

The resolution of a projector into an affinor and q-factors:—

$$r^k = \frac{1}{V}\sum_\lambda A^k_\lambda V^\lambda ; \; V\sum_k A^\nu_k r^k = \sum_{k\lambda} A^\nu_k A^k_\lambda V^\lambda$$
$$= \sum_\lambda A^\nu_\lambda . V^\lambda$$
$$V^\nu = V\sum_k A^\nu_k r^k + Vq^\nu = \sum_\lambda A^\nu_\lambda V^\lambda + Vq^\nu$$

We often denote the affinor part by $'V^\nu$. Therefore $V^\nu = 'V^\nu + Vq^\nu$; $'V^\nu = \sum_\lambda A^\nu_\lambda V^\lambda$ being the affinor part. .... (4.3)

In general a projector is an affinor when in *every suffix* its transvectant with q^ν or q_ν vanishes. Thus a projector $T^\lambda_{\mu\nu}$ is an affinor if

$$\left.\begin{aligned} \sum_\lambda q_\lambda T^\lambda_{\mu\nu} &= 0 \\ \sum_\mu q^\mu T^\lambda_{\mu\nu} &= 0 \\ \sum_\nu q^\nu T^\lambda_{\mu\nu} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.4)$$

Any projector can be expressed as a sum of products of affinors and factors q^ν, q_ν. The affinor part of a projector will be denoted by adjoining a dash. It is obtained by remembering that an affinor is unaltered by transvection with A^ν_μ whereas q^ν and q_ν vanish when transvected with A^ν_μ. Thus the affinor part of T^ν_μ is simply $\sum_{\rho\sigma} A^\rho_\mu A^\nu_\sigma T^\sigma_\rho$.

The metric in the E_4—Carry over the metric we introduced in the E^*_4 to the E_4 by means of the

correspondence set up above between E_4 and E^*_4. If l is the distance in the E^*_4 from the point of contact to the point V^ν,

$$l^2 = \frac{\omega^2}{V^2}\sum_{\lambda\mu}(G_{\lambda\mu} + q_\lambda q_\mu)\, V^\lambda V^\mu \quad \text{by } (3\cdot 5)$$

Write $$G_{\lambda\mu} + q_\lambda q_\mu = g_{\lambda\mu} \quad \ldots\ldots\ldots\ldots \quad (4\cdot 5)$$

So $$l^2 = \frac{\omega^2}{V^2}\sum_{\lambda\mu} g_{\lambda\mu} V^\lambda V^\mu$$

Now

$$\sum_\lambda q^\lambda g_{\lambda\mu} = \sum_\lambda q^\lambda (G_{\lambda\mu} + q_\lambda q_\mu) = q_\mu - q_\mu = 0.$$

Therefore $g_{\lambda\mu}$ is an affinor. The corresponding true affinor is

$$g_{ij} = \sum_{\lambda\mu} A^\lambda_i A^\mu_j g_{\lambda\mu}$$

Substituting from (4·3)

$$l^2 = \omega^2 \sum_{ij} g_{ij} r^i r^j.$$

Considering the immediate neighbourhood of the origin of the E_4 (the point of contact of the $E_4{}^*$), l can be written as ds

Comparing the infinitesimal equation

$$d\xi^i = \sum_\nu A^i_\nu dx^\nu,$$

with the finite equation

$$Vr^i = \sum_\nu A^i_\nu V^\nu,$$

we see that Vr^i or (since we are dealing with immediate neighbourhood of the origin,

$V = -\sum_{\nu} V^{\nu} q_{\nu} = \sum_{\nu} x^{\nu} q_{\nu} = \omega$) ωr^i reduces in the infinitesimal case to $d\xi^i$.

$$\left.\begin{aligned} &\text{So} && l^2 = \omega^2 \sum_{ii} g_{ij}\, r^i\, r^j \\ &\text{becomes} && ds^2 = \sum_{ij} g_{ij}\, d\xi^i\, d\xi^j \\ &\text{and} && ds^2 = \sum_{\lambda\,\mu} g_{\lambda\,\mu}\, dx^{\lambda}\, dx^{\mu} \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (4.6)$$

We identify g_{ij} with Riemann fundamental tensor of general relativity. So g_{ij} specifies gravitational field and $G^{\lambda\,\mu}$ specifies the gravitational field and the electro-magnetic field.

We readily prove that $g^{\lambda\,\mu} = G^{\lambda\,\mu} + q^{\lambda} q^{\mu}$

$$\sum_{\mu} g_{\lambda\,\mu} g^{\mu\,\nu} = A^{\nu}_{\lambda} \qquad \cdots (4 \cdot 7)$$

$$\sum_{j} g_{ij}\, g^{jk} = \delta^k_i$$

Covariant differentiation of a projector (of excess zero), can be defined by the equations

$$\nabla_{\mu}\ p = \frac{\partial p}{\partial x^{\mu}} \qquad \text{when } p \text{ is a scalar}$$

$$\nabla_{\mu} V^{\nu} = \frac{\partial V^{\nu}}{\partial x^{\mu}} + \sum_{\lambda} \Pi^{\nu}_{\lambda\,\mu} V^{\lambda} \qquad (4.8)$$

$$\nabla_{\mu} W_{\lambda} = \frac{\partial W_{\lambda}}{\partial x^{\mu}} - \sum \Pi_{\lambda\,\mu} W_{\nu}$$

Veblen in his projective relativity supposes that

$$\Pi^{\nu}_{\lambda\,\mu} = \Pi^{\nu}_{\mu\,\lambda}\,.$$

We do not suppose so here. $\Pi^{\nu}_{\lambda\mu}$ are a set of 125 functions of degree -1. They are not projectors just as in ordinary Einsteinian relativity the Christoffel symbols $\{^{\lambda\mu}_{\nu}\}$ are not tensors. In fact

$$
\left.\begin{aligned}
\Pi^{\nu'}_{\lambda'\mu'} &= \sum_{\lambda\mu\nu} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \cdot \frac{\partial x^{\mu}}{\partial x^{\mu'}} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Pi^{\nu}_{\lambda\mu} \\
&\qquad + \sum_{\nu} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \cdot \frac{\partial^2 x^{\nu}}{\partial x^{\lambda'} \partial x^{\mu'}} \\
\text{or} \qquad &= \sum_{\lambda\mu\nu} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \cdot \frac{\partial x^{\mu}}{\partial x^{\mu'}} \cdot \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Pi^{\nu}_{\lambda\mu} \\
&\qquad - \sum_{\lambda\mu} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \cdot \frac{\partial x^{\mu}}{\partial x^{\mu'}} \cdot \frac{\partial^2 x^{\nu'}}{\partial x^{\lambda} \partial x^{\mu}}
\end{aligned}\right\} \cdots (4\cdot 9)
$$

5. *The bivector $q_{\lambda\mu}$*:—

Define $q_{\lambda\mu}$ by

$$
\left.\begin{aligned}
q_{\lambda\mu} &= \tfrac{1}{2} \left(\frac{\partial q_{\lambda}}{\partial x^{\lambda}} - \frac{\partial q_{\lambda}}{\partial x^{\mu}} \right) \\
&= \tfrac{1}{2} \left(\nabla_{\lambda} q_{\mu} - \nabla_{\mu} q_{\lambda} \right)
\end{aligned}\right\} \cdots\cdots\cdots\cdots (5\cdot 1)
$$

Since $\sum q^{\lambda} q_{\lambda\mu} = 0$ .... $(5\cdot 2)$

$q_{\lambda\mu}$ is an affinor and it is skew $(q_{\lambda\mu} = -q_{\mu\lambda})$.

Hence it is called a bivector. We identify it (save for a constant factor) with the Electromagnetic bivector.

The Christoffel symbols :—

Write

$$\{{}^{\nu}_{\lambda\mu}\} = \tfrac{1}{2}\sum_{\rho} G^{\nu\rho}\left(\frac{\partial G_{\mu\rho}}{\partial x^{\lambda}} + \frac{\partial G_{\lambda\rho}}{\partial x^{\mu}} - \frac{\partial G_{\lambda\mu}}{\partial x^{\rho}}\right) \quad \ldots \quad (5\cdot 3)$$

and define

$$\Gamma^{k}_{ij} = \tfrac{1}{2}\sum_{t} g^{kt}\left(\frac{\partial g_{it}}{\partial \zeta^{j}} + \frac{\partial g_{jt}}{\partial \zeta^{i}} - \frac{\partial g_{ij}}{\partial \zeta^{t}}\right) \quad \ldots\ldots \quad (5\cdot 4)$$

Since

$G_{\lambda\mu} = g_{\lambda\mu} - q_{\lambda}\cdot q_{\mu}$; after some reduction (5 . 3) comes out as

$$\{{}^{\nu}_{\lambda\mu}\} = \sum_{ijk} A^{\nu}_{k} A^{i}_{\lambda} A^{j}_{\mu} \Gamma^{k}_{ij} + \sum_{k} A^{\nu}_{k} \frac{\partial^{2}\xi^{k}}{\partial x^{\lambda}\partial x^{\mu}} - q_{\lambda} q^{\cdot\nu}_{\mu} - q_{\mu} q^{\cdot\nu}_{\lambda} - \tfrac{1}{2} q^{\nu}\left(\frac{\partial q_{\mu}}{\partial x^{\lambda}} + \frac{\partial q_{\lambda}}{\partial x^{\mu}}\right) \quad \ldots \quad (5\cdot 5)$$

$$\Pi^{\nu}_{\lambda\mu} = \{{}^{\nu}_{\lambda\mu}\} + a\, q^{\nu} q_{\lambda\mu} + b\, q_{\lambda} q^{\nu}_{\cdot\mu} + c\, q_{\mu} q^{\nu}_{\cdot\lambda} \quad \ldots \quad (5\cdot 6)$$

where a, b, c are constants to be determined by the condition that $\nabla_{\mu} G_{\lambda\omega} = 0$ .... (5 . 7)

$$\left.\begin{array}{lll} (5\cdot 7) & \text{gives} & (a+b) = 0 \\ \text{Write} & q - 1 \text{ for } b \text{ or } -a \text{ and } p - 1 \text{ for } c & \\ (5\cdot 6) & \text{then becomes} & \end{array}\right\} (5\cdot 7\text{-}a)$$

$$\Pi^{\nu}_{\lambda\mu} = \left\{{\nu \atop \lambda\mu}\right\} - (q-1)\, q^{\nu} q_{\lambda\mu} + (p-1)\, q_{\lambda} q^{\nu}_{\cdot\mu} + (p-1)\, q_{\mu} q^{\nu}_{\cdot\lambda} \quad \ldots\ldots\ldots\ldots \quad (5\cdot 8)$$

In the symmetrical theories of Veblen, Hoffmann and Pauli, $p = 1$, $q = 1$. In the Einstein

and Mayer's five-dimensional unified theory, when projectised, q is undetermined while $p = 0$. In Schouten and van Dantzig theory up to the end of 1932 $p = 4$, $q = 2$. In Schouten's 1933 theory $q = 2$, p is arbitrary but satisfies $q^2 - 2pq + 2p > 0$.

The projector $S_{\lambda\,\mu}^{\cdot\,\cdot\,\nu}$ is defined as

$$\left.\begin{aligned} S_{\lambda\,\mu}^{\cdot\,\cdot\,\nu} &= \tfrac{1}{2}\left(\Pi^{\nu}_{\lambda\mu} - \Pi^{\nu}_{\lambda\mu}\right) \\ &= -(q-1)\,q_{\lambda\,\mu}\,q^{\nu} + \frac{p-q}{2}\,q_{\lambda}\,q_{\mu}^{\cdot\nu} \\ &\qquad - \frac{p-q}{2}\,q_{\mu}\,q_{\lambda}^{\cdot\nu} \end{aligned}\right\} \quad \cdots (5.9)$$

This gives

$$\left.\begin{aligned} \sum_{\nu\,\sigma} S_{\nu\,\sigma\,\lambda}\,q^{\nu}\,q^{\sigma} &= 0 \\ \sum_{\rho\,\nu} S_{\mu\,\rho\,\nu}\,q^{\rho}\,q^{\nu} &= 0 \\ \sum_{\rho\,\nu} S_{\rho\,\mu\,\nu}\,q^{\rho}\,q^{\nu} &= 0 \end{aligned}\right\} \quad \cdots\cdots (5.9\text{-}a)$$

6. Define projectors $P_{\cdot\lambda}^{\mu}$ and $Q_{\cdot\lambda}^{\nu}$ as

$$\left.\begin{aligned} P_{\cdot\lambda}^{\nu} &= \nabla_{\mu}\,x^{\nu} = \sum_{\mu} \Pi^{\nu}_{\lambda\,\mu}\,x^{\mu} + \delta^{\nu}_{\lambda} = -p\omega q_{\cdot\lambda}^{\nu} \\ Q_{\cdot\lambda}^{\nu} &= -\omega q q_{\cdot\lambda}^{\nu} \end{aligned}\right\} \quad \cdots (6\cdot 1)$$

(6.1) gives

$$\left.\begin{aligned} \nabla_{\mu}\,q^{\nu} &= -q q_{\cdot\mu}^{\nu} \\ \nabla_{\mu}\,q_{\nu} &= -q q_{\nu\,\mu} \\ \nabla_{\mu}\,A_{\lambda}^{\nu} &= -q\,(q_{\lambda}\,q_{\cdot\mu}^{\nu} + q^{\nu}\,q_{\lambda\,\mu}) \\ \nabla_{\mu}\,g_{\lambda\,\mu} &= -q\,(q_{\lambda}\,q_{\nu\,\mu} + q_{\nu}\,q_{\lambda\,\mu}) \end{aligned}\right\} \quad \cdots (6\cdot 2)$$

We determine an affine-connexion by means of a differential operator $\overset{R}{\nabla}_\mu$, which represents the affinor part of the projective covariant derivative of an affinor.

$$\overset{R}{\nabla}_\mu V^\nu = \sum_{\rho\,\sigma} A^\rho_\mu A^\nu_\sigma \nabla_\rho V^\sigma \quad , \quad \text{since } \sum V^\nu q_\nu = 0$$

$$\overset{R}{\nabla}_\mu W_\lambda = \sum_{\rho\,\pi} A^\rho_\mu A^\pi_\lambda \nabla_\rho W_\pi \quad , \quad \text{since } \sum_\nu W_\nu q_\nu = 0$$

The affinor (in Latin letters) corresponding to $\overset{R}{\nabla}_\mu V^\nu$ is

$$\overset{R}{\nabla}_j V^\kappa = \sum_{\rho\,\sigma} A^\rho_j A^\kappa_\sigma \nabla_\rho V^\sigma$$

Similarly
$$\overset{R}{\nabla}_j W_i = \sum_{\rho\,\pi} A^\rho_j A^\pi_i \nabla_\rho W_\pi$$

We extend this affine connexion to a projective one by the additional conditions $\overset{R}{\nabla}_\mu q^\nu = 0$ and $\overset{R}{\nabla}_\mu q_\nu = 0$.

Let V^ν be any contravariant point of excess zero and so of degree one. Then

$$\left.\begin{aligned} \overset{R}{\nabla}_\mu V^\nu &= \frac{\partial V^\nu}{\partial x^\mu} + \sum_\lambda \overset{R}{\Pi}{}^\nu_{\lambda\mu} V^\lambda \\ \text{where } \overset{R}{\Pi}{}^\nu_{\lambda\mu} &= \left\{ {}^{\,\nu}_{\lambda\mu} \right\} + q^\nu q_{\lambda\mu} - q_\lambda q^\nu_{\cdot\mu} - q_\mu q^\nu_{\cdot\mu} \end{aligned}\right\} (6\cdot3)$$

are the parameters of the covariant differentiation

of the Riemannian geometry written in homogeneous coordinates.

Applying this $\qquad \overset{R}{\nabla}_j g_{ik} = 0$

Also $\qquad \overset{R}{S}_{\lambda\cdot\mu}^{\cdot\nu} = \frac{1}{2}\left[\overset{R}{\Pi}_{\lambda}{}^{\nu}_{\mu} - \overset{R}{\Pi}^{\nu}_{\mu\lambda}\right] = q_{\lambda\mu} q^{\nu}$

If F_{ij} is the electromagnetic six-vector then

$F_{ij} = \frac{\partial \varphi_j}{\partial \zeta_i} - \frac{\partial \varphi_i}{\partial \zeta_j}$ where φ_1, φ_2, φ_3 are the components of the magnetic potential-vector and φ_4 is the electric potential. We identify F_{ij} as

$$F_{ij} = \frac{qc}{k} q_{ij} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6\cdot4)$$

where c is the velocity of light and k is a constant of dimensions $M^{-1/2}L^{1/2}$

(6·4) gives in homogeneous coordinates

$$F_{\lambda\nu} = \frac{qc}{k} q_{\lambda\mu}, \text{ that is, } \frac{\partial \varphi_\mu}{\partial x^\lambda} - \frac{\partial \varphi_\lambda}{\partial x^\mu} = \frac{qc}{2k}\left(\frac{\partial q_\mu}{\partial x^\lambda} - \frac{\partial q_\lambda}{\partial x^\mu}\right)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots \quad (6\cdot5)$$

where φ_λ is the electromagnetic potential.

In Einstein's general relativity we have the Riemann Tensor

$$K_{pqr}^{\cdots s} = \frac{\partial}{\partial x^r}\begin{Bmatrix} pq \\ s \end{Bmatrix} - \frac{\partial}{\partial x^q}\begin{Bmatrix} pr \\ s \end{Bmatrix} + \sum_t \begin{Bmatrix} pq \\ t \end{Bmatrix}\begin{Bmatrix} tr \\ s \end{Bmatrix} - \sum_t \begin{Bmatrix} pr \\ t \end{Bmatrix}\begin{Bmatrix} qt \\ s \end{Bmatrix}$$

This referred to homogeneous coordinates gives

$$K_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} = \frac{\partial \overset{R}{\Pi}{}^{\nu}_{\lambda\,\omega}}{\partial x^{\mu}} - \frac{\partial \overset{R}{\Pi}{}^{\nu}_{\lambda\,\mu}}{\partial x^{\omega}} + \sum_{\rho} \overset{R}{\Pi}{}^{\nu}_{\rho\,\mu} \overset{R}{\Pi}{}^{\rho}_{\lambda\,\omega} - \sum_{\rho} \overset{R}{\Pi}{}^{\nu}_{\rho\,\omega} \overset{R}{\Pi}{}^{\rho}_{\lambda\,\mu} \qquad \ldots\ldots\ldots\ldots \quad (6.6)$$

From $\Pi^{\nu}_{\lambda\,\mu}$ we form the *projective curvature tensor*

$$N_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} = \frac{\partial \Pi^{\nu}_{\lambda\,\omega}}{\partial x^{\mu}} - \frac{\partial \Pi^{\nu}_{\lambda\,\mu}}{\partial x^{\omega}} + \sum_{\rho} \Pi^{\nu}_{\rho\,\mu} \Pi^{\rho}_{\lambda\,\omega} - \sum_{\rho} \Pi^{\nu}_{\rho\,\omega} \Pi^{\rho}_{\lambda\,\mu} \qquad \ldots\ldots\ldots\ldots \quad (6.7)$$

From (6.6) and (6.7) we get

$$\left.\begin{aligned} N_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} - K_{\omega\mu\lambda}^{\cdot\,\cdot\,\cdot\,\nu} = {} & q\, q_{\lambda} \overset{R}{\nabla}_{\mu} q^{\nu}_{\cdot\omega} - q\, q^{\nu} \overset{R}{\nabla}_{\mu} q_{\lambda\,\omega} \\ & + p\, q_{\omega} \overset{R}{\nabla}_{\mu} q^{\nu}_{\cdot\lambda} - q\, q_{\lambda} \overset{R}{\nabla}_{\omega} q^{\nu}_{\cdot\mu} \\ & + q\, q^{\nu} \overset{R}{\nabla}_{\omega} q_{\lambda\,\mu} - p\, q_{\mu} \overset{R}{\nabla}_{\mu} q^{\nu}_{\cdot\lambda} \\ & + q^{2}\, q^{\nu}_{\cdot\mu}\, q_{\lambda\,\omega} - 2p\, q_{\omega\,\mu} q^{\nu}_{\cdot\lambda} \\ & + pq(-q^{\nu} q_{\omega} q_{\rho\,\mu} q^{\rho}_{\cdot\lambda} + q_{\lambda} q_{\mu} q^{\nu}_{\cdot\rho} q^{\rho}_{\cdot\omega} \\ & + q_{\lambda} q_{\mu} q^{\nu}_{\cdot\rho} q^{\rho}_{\cdot\omega} + q^{\nu} q_{\mu} q_{\rho\,\omega} q^{\rho}_{\cdot\lambda} \\ & - q_{\lambda} q_{\omega} q^{\nu}_{\cdot\rho} q^{\rho}_{\cdot\mu}) \end{aligned}\right\} (6.8)$$

Forming the generalized contracted Riemann tensor

$$\left.\begin{aligned} & N_{\mu\lambda} = \sum_{\rho} N_{\rho\mu\lambda}^{\cdot\,\cdot\,\cdot\,\rho} \;;\quad K_{\mu\lambda} = \sum_{\rho} K_{\rho\mu\lambda}^{\cdot\,\cdot\,\cdot\,\rho} \\ & \text{and the scalars} \\ & N = \sum_{\mu\lambda} G^{\mu\lambda} N_{\mu\lambda} \text{ and } K = \sum_{\lambda\mu} g^{\mu\lambda} K_{\mu\lambda} \end{aligned}\right\} \ldots\ldots \quad (6.9)$$

we get

$$\left.\begin{aligned} N_{\mu\lambda} - K_{\mu\lambda} = &- q\, q_\lambda \sum_\rho \overset{R}{\nabla}_\rho\, q^{\rho}_{.\mu} \\ &- pq_\mu \sum_\rho \overset{R}{\nabla}_\rho\, q^{\rho}_{.\lambda} - pqq_\lambda q_\mu \sum_{\rho\,\sigma} q^{\rho\sigma} q_{\rho\sigma} \\ &+ (pq - 2p - q^2) \sum_\rho q_{\rho\mu} q^{\rho}_{.\mu} \end{aligned}\right\} \cdots \quad (6.9\text{-}a)$$

and

$$N - K = (2pq - 2p - q^2) \sum_{\rho\,\sigma} q_{\rho\sigma} q^{\rho\sigma} \quad \cdots\cdots (6.9\text{-}b)$$

SECOND LECTURE

FIELD-EQUATIONS AND THE IDENTITIES

1. *The variation Principle and the Field-Equations*:—The simplest variational principle $\delta I = 0$ where $I = \int N \sqrt{G}\, dx^0\, dx^1\, dx^2\, dx^3\, dx^4 \quad \ldots \quad (1.1)$ where N is the projective scalar curvature gives the field equations. In variation, $G_{\lambda\mu}$ are to be varied while the x^ν are to be kept constant. Working out the variation we get

$$\delta I = \sum_{\lambda\mu} \int \sqrt{G}. \Big[K_{\lambda\mu} - \tfrac{1}{2} K g_{\lambda\mu} + (q^2 - 2pq + 2p) \left\{ \tfrac{1}{2} g_{\lambda\mu} \sum_{\rho\sigma} q_{\rho\sigma} q^{\rho\sigma} - 2 \sum_{\rho} q^{\cdot\rho}_{\lambda} q_{\mu\rho} + 2 q_\lambda \sum_{\rho} \overset{R}{\nabla}_\rho q^{\cdot\rho}_{\mu} \right\} \Big] \delta G^{\lambda\mu} . d\tau \qquad \ldots\ldots\ldots\ldots (1.2)$$

where $\qquad d\tau = dx^0\, dx^1\, dx^2\, dx^3\, dx^4.$

So the variational equations are

$$K_{\lambda\mu} - \tfrac{1}{2} K g_{\lambda\mu} + (q^2 - 2pq + 2p) \left\{ \tfrac{1}{2} g_{\lambda\mu} \sum_{\rho\sigma} q_{\rho\sigma} q^{\rho\sigma} - 2 \sum_{\rho} q^{\cdot\rho}_{\lambda} q_{\mu\rho} + 2 q_\lambda \sum_{\rho} \overset{R}{\nabla}_\rho q^{\cdot\rho}_{\mu} \right\} = 0 \qquad \ldots\ldots\ldots\ldots (1.3)$$

Subtract this from the equation obtained by interchanging λ and μ. We get

$$q_\lambda \sum_\rho \overset{R}{\nabla}_\rho q_\mu^{\cdot\rho} - q_\mu \sum_\rho \overset{R}{\nabla}_\rho q_\lambda^{\cdot\rho} = 0 \quad (\lambda,\mu = 0, 1, 2, 3, 4)$$

These equations can only be satisfied if

$$\sum_\rho \overset{R}{\nabla}_\rho q_\lambda^{\cdot\rho} = 0 = B_\lambda \text{ say } (\lambda = 0, 1, 2, 3, 4) \quad \ldots \quad (1\cdot 4)$$

(7·3) then reduces to the equation symmetrical in λ and μ

$$K_{\lambda\mu} - \tfrac{1}{2}Kg_{\lambda\mu} + (q^2 - 2q + 2p)$$

$$\left\{\tfrac{1}{2}g_{\lambda\mu} \sum_{\rho\sigma} q_{\rho\sigma}q^{\rho\sigma} - 2\sum_\rho q_\lambda^{\cdot\rho} q_{\mu\rho}\right\} = 0 \quad \ldots\ldots \quad (1\cdot 5)$$

The Einsteinian field-equations of Gravitation and Electromagnetism in empty space are

$$K_{ij} - \tfrac{1}{2} Kg_{ij} = \varkappa E_{ij}$$

where $\varkappa = 8\pi\mu$ where μ is the Newtonian gravitational potential due to one erg of matter at a distance of 1 cm.,

K_{ij} is the contracted Riemann Tensor

K is the scalar curvature

and E_{ij} is the Energy tensor

In empty space

$$E_{ij} = \frac{1}{4c^2} g_{ij} \sum_{kl} F_{kl} F^{kl} - \frac{1}{c^2} \sum F_i^{\cdot k} F_{jk}$$

where F_{pq} is the electromagnetic six vector.

(If d_x, d_y, d_z is the electric vector

h_x, h_y, h_z is the magnetic vector

$$F_{14} = \frac{1}{c} d_x,\ F^{41} = c\, d_x,\ F_{23} = \frac{1}{c} h_x,\ F^{23} = c^3 h_x$$

and so on).

Thus the field equations are

$$\left.\begin{aligned} & K_{ij} - \tfrac{1}{2} K g_{ij} + \frac{\varkappa}{c^2}\Big(\sum_k F_i^{\cdot k} F_{jk} \\ & \qquad - \tfrac{1}{4} g_{ij} \sum_{kl} F_{kl}\ F^{kl}\Big) = 0 \\ & \text{or in homogeneous coordinates} \\ & K_{\lambda\mu} - \tfrac{1}{2} K g_{\lambda\mu} + \frac{\varkappa q^2}{k^2}\Big(\sum_\rho q_{\cdot\lambda}^{\rho} q_{\mu\rho} \\ & \qquad - \tfrac{1}{4} g_{\lambda\mu} \sum_{\rho\sigma} q_{\rho\sigma} q^{\rho\sigma}\Big) = 0 \end{aligned}\right\} \quad \dots\dots\dots \quad (1.6)$$

with help of (6.4) and (6.5) of the last lecture.

This must be identified with (1.5). So we must have

$$-2\,(q^2 - 2pq + 2p)\,k^2 = \varkappa q^2 \quad \dots\dots\dots \quad (1.7)$$

$$\left.\begin{aligned} & \text{(1.4) gives} \quad \sum_j \overset{R}{\nabla}_j F_i^{\cdot j} = 0 \text{ or } \triangle_{iv}(F^{ij}) = 0. \\ & \text{This is the Maxwell's tetrad of equations} \\ & \frac{\partial d_x}{\partial x} + \frac{\partial d_y}{\partial y} + \frac{\partial d_z}{\partial z} = 0\,;\ -\frac{\partial d_x}{\partial t} + \frac{\partial h_z}{\partial y} - \frac{\partial h_y}{\partial z} = 0 \\ & \qquad\qquad\qquad\qquad\qquad \text{and so on.} \end{aligned}\right\} (1.8)$$

The other tetrad of Maxwell's equations are equivalent to the fact that $q_{\lambda\mu}$ is expressible as

$$q_{\lambda\mu} = \tfrac{1}{2}\left(\frac{\partial q_\lambda}{\partial x^\mu} - \frac{\partial q^\mu}{\partial x^\lambda}\right) \quad \text{by (5.1) of last lecture.}$$

2. *The Equations of Dirac*:—Dirac's numbers for projective theory are defined by the equations

$$\alpha^{(\lambda}\alpha^{\kappa)} = G^{\lambda\kappa} \qquad \ldots\ldots\ldots\ldots \quad (2.1)$$

and by using these numbers the Dirac-equation in Euclidian space can be written

$$\alpha^\mu\left(\frac{h}{i}\frac{\partial}{\partial\mu} - \frac{e}{c}\varphi_\mu + mcq_\mu\right)\psi = 0 \quad \ldots\ldots\ldots \quad (2\cdot 2)$$

In Riemannian space this equation is

$$\alpha^\mu\left(\frac{h}{i}\overset{R}{\nabla}_\mu - \frac{e}{c}\varphi_\mu + mcq_\mu\right)\psi = 0 \quad \ldots\ldots\ldots \quad (2\cdot 3)$$

$$\left.\begin{aligned} \text{where } \overset{R}{\nabla}_\mu &= \frac{\partial}{\partial x_\mu} + \overset{R}{\Lambda}_\mu \\ \overset{R}{\Lambda}_\mu &= -\tfrac{1}{4}\overset{R}{\Pi}{}^\kappa_{\mu\lambda}\,\alpha^\lambda\alpha_\kappa + \tfrac{1}{4}\alpha^\kappa\frac{\partial}{\partial x_\mu}\alpha_\kappa \end{aligned}\right. \qquad (2\cdot 4)$$

Now in projective theory we have to use instead of $\overset{R}{\nabla}_\mu$

$$\left.\begin{aligned} \nabla_\mu &= \frac{\partial}{\partial x_\mu} + \Lambda_\mu \\ \Lambda_\mu &= -\tfrac{1}{4}\Pi^\kappa_{\mu\lambda}\,\alpha^\lambda\alpha_\kappa + \tfrac{1}{4}\alpha^\kappa\frac{\partial}{\partial x_\mu}\alpha_\kappa \end{aligned}\right\} \quad (2\cdot 5)$$

and the question arises whether the use of ∇_μ instead of $\overset{R}{\nabla}_\mu$ gives use to additional terms in the equations.

Now $\alpha^\mu(\nabla_\mu - \overset{R}{\nabla}_\mu) = \frac{1}{4}(p-2q)\alpha^{[\mu\lambda\kappa]}q_\lambda q_{\mu\kappa}$ (2·6)

Hence if we impose the condition that

The Dirac equation of projective relativity is identical with the ordinary Dirac equation and contains no additional terms; we have then

$$p - 2q = 0 \quad \text{..................} \quad (2.7)$$

3. *The variational equation in the case of a current.*

In ordinary theory the term with the current in the Maxwell equation (7.8), results, if we take instead of N a world function $\overset{\circ}{M} + K$ where

$$\overset{\circ}{M} = \sqrt{g}\,\frac{i\varkappa}{c}\,\frac{h}{i}\bar{\psi}\,\eta\,\alpha^\mu\,\overset{R}{\nabla}_\mu\psi \quad \text{...........} \quad (3.1)$$

This function is "practical real," that is, the imaginary part is a divergence. This is necessary because otherwise the result could not be real. In projective theory we have naturally to take instead of $\overset{\circ}{M}$

$$\overset{\circ}{M} = \sqrt{G}\,\frac{i\varkappa}{c}\,\frac{h}{i}\bar{\psi}\,\eta\,\alpha^\mu\,\nabla_\mu\psi \quad \text{...........} \quad (3.2)$$

Hence we have to impose the condition *that M is practical real* .... (3.3)

But this condition is identically satisfied if $\omega^2 > 0$. It is remarkable that for $\omega^2 < 0$ we find as a necessary and sufficient condition

$$p - 2q = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

Hence (3.3) implies (2.7)

We now take $\omega > 0$ and $- - - - +$ as signature of $G_{\lambda\kappa}$. The variational equation

$$\delta \int (M + N)\, dx^0\, dx^1\, dx^2\, dx^3\, dx^3 = 0 \quad \ldots\ldots \quad (3.4)$$

gives

1. The equations of energy and impulse

$$\left\{ \begin{array}{l} \text{of gravitation} \\ \text{of Electromagnetic field} \\ \text{and of material waves} \end{array} \right.$$

2. The equation of Maxwell

$$\overset{R}{\nabla}_j F_i^{\cdot j} = es_i + \frac{p - 2q}{2q} \frac{h\varkappa}{i} \overset{R}{\nabla} k\, (\bar{\psi} \alpha_{[k} \alpha_{i]} \alpha^0 \psi) \quad (3.5)$$

in which equation es_i is the current-vector,

$$s_\lambda \text{ being} = - s q_\lambda ;\ (s = q^\lambda s_\lambda) \quad \ldots\ldots\ldots \quad (3.6)$$

The additional term in (3.5) vanishes only if we impose the condition (2.7).

*4. *A variational theorem*—Emmy Nöether,† has proved a general theorem to the effect that:

If F is a function of n quantities f, which are themselves functions of the m coordinates

$$(x^0, x^1, \ldots\ldots\ldots\ldots, x^{m-1})$$

and their derivatives and if the integral

$$\int F \,.\, dx^0 \,.\, dx^1 \ldots\ldots\ldots\ldots\, dx^{m-1}$$

is invariant with respect to arbitrary transformations of the coordinates $(x^0, x^1, \ldots\ldots\ldots, x^{m-1})$, then in the system of the n Lagrangian differential equations, which belong to the variational problem

$$\delta \int F \; dx^0 \ldots\ldots\ldots\ldots\, dx^{m-1} = 0$$

there are always m, which are a consequence of the $n - m$ others, in the sense, that between the n quantities f and their total differential coefficients with respect to $x^0, x^1, \ldots\ldots\ldots\ldots, x^{m-1}$, m linearly independent relations are identically satisfied. The best method of finding these identical relations in any particular problem is one due to Klein,‡

*Sections 4, 5 and 6 contain in a modified form, the results of my paper: "Identical relations between the field equations in the general field theory of Schouten and van Dantzig", in the *phil. Mag.*, Series 7, vol. xxii, p. 950.

†*Göttingen Nachrichten*, p. 236 (1918).

‡*Gött. Nach.*, p. 469 (1917).

which is based on Lie's theory of infinitesimal transformations.

Let us try to work out the above theorem for the variational integral (1.1)

$$\int N\sqrt{G}\,d\tau$$

where N is the projective scalar curvature and

$$d\tau = dx^0\, dx^1\, dx^2\, dx^3\, dx^4.$$

By (1.2) we get

$$\delta\int N\sqrt{G}d\tau = \sum_{\lambda\mu}\int\sqrt{G}.P_{\lambda\mu}.\delta G^{\lambda\mu}.d\tau \;\ldots\ldots\ldots\; (4.1)$$

where

$$P_{\lambda\mu} \equiv \Big[K_{\lambda\mu} - \tfrac{1}{2}Kg_{\lambda\mu} + (q^2 - 2pq + 2p)$$

$$\Big\{\tfrac{1}{2}g_{\lambda\mu}\sum_{\rho\sigma} q_{\rho\sigma}q^{\rho\sigma} + 2q_\lambda\sum_{\rho}\overset{R}{\nabla}_\rho q^{\cdot\rho}_{\mu} - 2\sum_{\rho} q^{\rho}_{\cdot\lambda}q_{\mu\rho}\Big\}\Big] \quad (4.2)$$

5. *To find the value of* $\delta G^{\lambda\mu}$.

Compare $G^{\lambda\mu}$ and $G^{\lambda\mu} + \delta G^{\lambda\mu}$. Since they correspond to a transformation of coordinates, we have

$$G^{\lambda\mu} + \delta G^{\lambda\mu} = G^{\alpha\beta}\frac{\partial(x^\lambda + \delta x^\lambda)}{\partial x^\alpha}\cdot\frac{\partial(x^\mu + \delta x^\mu)}{\partial x^\beta}$$

$$= G^{\alpha\beta}\frac{\partial x^\lambda}{\partial x^\alpha}\cdot\frac{\partial x^\mu}{\varrho x^\beta} + G^{\alpha\beta}\frac{\partial x^\lambda}{\partial x^\beta}\cdot\frac{\partial(\delta x^\mu)}{\partial x^\beta}$$

$$+ G^{\alpha\beta}\frac{\partial x^\mu}{\partial x^\beta}\cdot\frac{\partial(\delta x^\lambda)}{\partial x^\alpha}$$

$$= G^{\lambda\mu} + G^{\lambda\beta}\cdot\frac{(\partial p^\mu)}{\partial x^\mu}\, G^{\alpha\beta}\frac{\partial(p^\lambda)}{\partial x^\alpha}$$

where $$p^\lambda = \delta x^\lambda \text{ and } p^\mu = \delta x^\mu .$$

Hence $$\delta G^{\lambda\mu} = G^{\lambda\beta}\frac{\partial(p^\mu)}{\partial x^\beta} + G^{\alpha\mu}\cdot\frac{\partial(p^\lambda)}{\partial x^\alpha}$$

We keep $d\tau$ fixed in comparison and compare the values at x^α in both the systems. We have to subtract the change

$$\left\{\delta x^\alpha \frac{\partial G^{\lambda\mu}}{\partial x^\alpha}\right\} \text{ of } G^{\lambda\mu} \text{ in the distance } \delta x^\alpha .$$

Hence

$$\delta G^{\lambda\mu} = G^{\lambda\alpha}\frac{\partial(p^\mu)}{\partial x^\alpha} + G^{\alpha\mu}\frac{\partial(p^\lambda)}{\partial x^\alpha} - \frac{\partial G^{\lambda\mu}}{\partial x^\alpha}\, p^\alpha \qquad (5\cdot1)$$

where $$p^\alpha = \delta x^\alpha$$

6. Using (5 . 2), (4 . 1) becomes

$$\int \sum_{\lambda\mu} \sqrt{G} \,.\, P_{\lambda\mu} \,.\, \delta G^{\lambda\mu}\, d\tau$$

$$= \sum_{\lambda\mu\alpha} \int \sqrt{G} \,.\, P_{\lambda\mu}\Big(G^{\lambda\alpha}\frac{\partial p^\mu}{\partial x^\alpha} + G^{\mu\alpha}\frac{\partial p^\lambda}{\partial x^\alpha}$$

$$\frac{\partial G^{\lambda\mu}}{\partial x^\alpha} \,.\, p^\alpha \Big) d\tau$$

We integrate this by parts, supposing the p's and their first and second derivatives to vanish at

the boundary. We get

$$\sum_{\alpha\mu\lambda}\int\Big[\frac{\partial}{\partial x^{\alpha}}\Big\{P_{\lambda\mu}\sqrt{G}\,.\,G^{\lambda\alpha}\Big\}p^{\mu}+\frac{\partial}{\partial x^{\alpha}}\Big\{P_{\lambda\mu}\sqrt{G}\,.\,G^{\alpha\mu}\Big\}p^{\lambda}+\,.\,\frac{\partial G^{\lambda\mu}}{\partial x^{\alpha}}p^{\alpha}\,.\,P_{\lambda\mu}\sqrt{G}\Big]d\tau=0$$

Interchanging the dummy suffixes in each of the first two terms on the left we have

$$\sum_{\lambda\mu\alpha}\int\Big[\frac{\partial}{\partial x^{\mu}}\Big\{P_{\lambda\alpha}\sqrt{G}\,.\,G^{\lambda\mu}\Big\}+\frac{\partial}{\partial x^{\lambda}}\Big\{P_{\alpha\mu}\sqrt{G}\,.\,G^{\lambda\mu}\Big\}+\frac{\partial G^{\lambda\mu}}{\partial x^{\alpha}}P_{\lambda\mu}\sqrt{G}\Big]p^{\alpha}d\tau=0$$

Since p^{α} is arbitrary, its coefficients in this equation must vanish. Hence

$$\sum_{\lambda\mu}\Big[\frac{\partial}{\partial x^{\mu}}\Big\{P_{\lambda\mu}.G^{\lambda\mu}\sqrt{G}\Big\}+\frac{\partial}{\partial x^{\lambda}}\Big\{P_{\alpha\mu}\sqrt{G}\,.\,G^{\lambda\mu}\Big\}+\frac{\partial G^{\lambda\mu}}{\partial x^{\alpha}}P_{\lambda\mu}\sqrt{G}\Big]=0$$

$$(\alpha=0,1,2,3,4).$$

Interchanging λ and μ in the second term we get

$$\sum_{\lambda\mu}\Big[\frac{\partial}{\partial x^{\mu}}\Big\{(P_{\lambda\alpha}+P_{\alpha\lambda})\sqrt{G}\,.\,G^{\lambda\mu}\Big\}+\frac{\partial G^{\mu\lambda}}{\partial x^{\alpha}}P_{\lambda\mu}\sqrt{G}\Big]=0\;\ldots\ldots\;(6.1)$$

$$(\alpha=0,1,2,3,4).$$

Substitute in (6·1)

$$\frac{\partial G^{\lambda\mu}}{\partial x^{\alpha}} = -\sum_{\sigma}\begin{Bmatrix}\mu\\ \alpha\sigma\end{Bmatrix}G^{\lambda\sigma} - \sum_{\sigma}\begin{Bmatrix}\lambda\\ \alpha\sigma\end{Bmatrix}G^{\mu\sigma} \quad \ldots\ldots \quad (6\cdot 2)$$

a formula which can be proved in the same manner as the corresponding formula in ordinary tensor-analysis and put

$$X^{\mu\cdot}_{\cdot\alpha} = \tfrac{1}{2}\left(P^{\mu\cdot}_{\cdot\alpha} + P^{\cdot\mu}_{\alpha\cdot}\right) \quad \ldots\ldots\ldots\ldots \quad (6\cdot 3)$$

Then we have

$$\left.\begin{array}{l} \sum_{\mu}\left[2\frac{\partial}{\partial x^{\mu}}\left(X^{\mu\cdot}_{\cdot\alpha}\right)\cdot\sqrt{G} + 2X^{\mu\cdot}_{\cdot\alpha}\frac{\partial\sqrt{G}}{\partial x^{\mu}}\right] \\ \qquad\qquad \sqrt{G}\left[\sum_{\sigma\mu}\begin{Bmatrix}\mu\\ \alpha\sigma\end{Bmatrix}P^{\sigma\cdot}_{\cdot\mu} + \begin{Bmatrix}\lambda\\ \alpha\sigma\end{Bmatrix}P^{\cdot\sigma}_{\lambda\cdot}\right] \end{array}\right\} = 0 \quad (6\cdot 4)$$

$(\alpha = 0, 1, 2, 3, 4).$

Changing the dummy suffixes and remembering that

$$\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial x^{\mu}} = \sum_{\sigma}\begin{Bmatrix}\sigma\\ \mu\sigma\end{Bmatrix} \qquad (6\cdot 5)$$

which too can be proved as in ordinary tensor-calculus, we get

$$2\sqrt{G}\cdot\sum_{\mu}\left[\frac{\partial}{\partial x^{\mu}}X^{\mu\cdot}_{\cdot\alpha} + \sum_{\sigma}\begin{Bmatrix}\sigma\\ \mu\sigma\end{Bmatrix}X^{\mu\cdot}_{\cdot\alpha} - \sum_{\sigma}\begin{Bmatrix}\sigma\\ \alpha\mu\end{Bmatrix}X^{\mu\cdot}_{\cdot\sigma}\right] = 0$$

$(\alpha = 0, 1, 2, 3, 4)$

Or

$$\sum_{\mu}\left[\frac{\partial}{\partial x^{\mu}}(X^{\mu\cdot}_{\cdot\alpha})+\sum_{\sigma}\left\{{\sigma \atop \mu\sigma}\right\}X^{\mu\cdot}_{\cdot\alpha}\right]-\sum_{\sigma\mu}\left\{{\sigma \atop \alpha\mu}\right\}X^{\mu\cdot}_{\cdot\sigma}=0 \qquad \ldots\ldots\ldots\ldots \quad (6.6)$$

$(\alpha = 0, 1, 2, 3, 4)$

Now by means of (4.8) and (5.8) of the first lecture we have

$$\left.\begin{aligned}
(i)\quad & \nabla_{\mu}X^{\mu\cdot}_{\cdot\alpha} = \frac{\partial X^{\mu\cdot}_{\cdot\alpha}}{\partial x^{\mu}} - \sum_{\sigma\mu}\Pi^{\sigma}_{\alpha\mu}X^{\mu\cdot}_{\cdot\sigma} + \sum_{\sigma\mu}\Pi^{\mu}_{\sigma\mu}X^{\sigma\cdot}_{\cdot\alpha}\\
(ii)\quad & \Pi^{\mu}_{\sigma\mu} = \left\{{\mu \atop \sigma\mu}\right\}\\
(iii)\quad & \Pi^{\sigma}_{\alpha\mu} = \left\{{\sigma \atop \alpha\mu}\right\} - (q-1)q^{\sigma}q_{\alpha\mu} + (q-1)\\
& \qquad q_{\alpha}q^{\sigma}_{\cdot\mu} + (p-1)q_{\mu}q^{\sigma}_{\cdot\alpha}
\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \quad (6.7)$$

Making use of (6.7 *i*) in (6.6) we have

$$\sum_{\mu\sigma}\Big[\nabla_{\mu}X^{\mu\cdot}_{\cdot\alpha} + \Pi^{\sigma}_{\alpha\mu}X^{\mu\cdot}_{\cdot\sigma} + \left\{{\sigma \atop \mu\sigma}\right\}X^{\mu\cdot}_{\cdot\alpha} - \Pi^{\mu}_{\sigma\mu}X^{\sigma\cdot}_{\cdot\alpha} - \left\{{\sigma \atop \alpha\mu}\right\}X^{\mu\cdot}_{\cdot\sigma}\Big] = 0 \qquad \ldots\ldots\ldots\ldots \quad (6.8$$

$(\alpha = 0, 1, 2, 3, 4)$

Since

$$\Pi^{\mu}_{\sigma\mu}X^{\sigma\cdot}_{\cdot\alpha} = \left\{{\mu \atop \sigma\mu}\right\}X^{\sigma\cdot}_{\cdot\alpha} = \left\{{\sigma \atop \mu\sigma}\right\}X^{\mu\cdot}_{\cdot\alpha}$$

we see that the third and the fourth terms in (6.8) cancel out. With the help of (6.7 *iii*) we finally arrive at the identities between the field-equations in the form

$$\sum_{\mu\sigma}\Big[\nabla_{\mu}X^{\mu\cdot}_{\cdot\alpha} - \{(q-1)q^{\sigma}q_{\alpha\mu} - (q-1)q_{\alpha}q^{\sigma}_{\cdot\mu} - (p-1)q_{\mu}q^{\sigma}_{\cdot\alpha}\}X^{\mu\cdot}_{\cdot\sigma}\Big] = 0 \qquad \ldots\ldots\ldots\ldots \quad (6.9)$$

$(\alpha = 0, 1, 2, 3, 4)$

Changing the dummy suffix σ into λ we have the required identities as

$$\sum_{\lambda\mu}[\nabla_\mu X^{\mu\cdot}_{\cdot a} - \{(q-1)q^\lambda q_{a\mu} - (q-1)q_a q^\lambda_{\cdot\mu} - (p-1)q_\mu q^\lambda_{\cdot a}\}X^{\mu\cdot}_{\cdot\lambda}] = 0 \quad \ldots\ldots\ldots\ldots \quad (6.10)$$

$$(a = 0, 1, 2, 3, 4)$$

Third Lecture*

Verification of the Identities and Connection Between the General Relativity Theory and the Present Theory

1. *Verification of the identities*—We have

$$X_{\lambda\mu} = N_{\lambda\mu} - \tfrac{1}{2} N G_{\lambda\mu} + (3pq - q^2 - 2p) \sum_{\rho} q_{\lambda}^{\cdot\rho} q_{\mu\sigma}$$

$$- (p + q)\, q_{\lambda} \sum \overset{R}{\nabla}_{\rho} q_{\mu}^{\cdot\rho} + (q^2 - 2pq + 2p)$$

$$\left[q_{\lambda} \sum_{\rho} \overset{R}{\nabla}_{\rho} q_{\mu}^{\cdot\rho} + q_{\mu} \sum_{\rho} \overset{R}{\nabla}_{\rho} q_{\lambda}^{\cdot\rho} \right] \dots \quad (1.1)$$

This can be written as

$$X_{\lambda\mu} = Z_{\lambda\mu} + (q^2 - 2pq + 2p)\, Y_{\lambda\mu} \quad \dots\dots\dots\dots \quad (1.2)$$

where

$$Z_{\lambda\mu} = N_{\lambda\mu} - \tfrac{1}{2} N G_{\lambda\mu} + (-q^2 + 3pq - 2p)$$

$$\sum_{\rho} q_{\lambda}^{\cdot\rho} q_{\mu\rho} - (p + q) q^{\lambda} \sum_{\rho} \overset{R}{\nabla} q_{\mu}^{\cdot\rho} \quad \dots\dots \quad (1.3)$$

*Contents of this lecture excepting the last section are the modified form of some portions of my paper "Identities between field-equations in the general field-theory of Schouten and van Dantzig" published in the *Journal of the Indian Mathematical Society*, New Series, Vol. II, No. 5.

and $$Y_{\lambda\mu} = q^{\lambda} \sum_{\rho} \overset{R}{\nabla}_{\rho} q^{\cdot\rho}_{\mu} + q_{\mu} \sum_{\rho} \overset{R}{\nabla}_{\rho} q^{\cdot\rho}_{\lambda} . \qquad (1.4)$$

Changing α into λ, λ into σ and substituting from (1.2) the left side of the identities breaks up into the sum of two parts namely,

$$\sum_{\mu} \nabla_{\mu} Z^{\cdot\mu}_{\lambda} - (q-1) \sum_{\mu\sigma} Z^{\cdot\mu}_{\sigma\cdot} q^{\sigma} q_{\lambda\mu} + (q-1) q_{\lambda} \sum_{\sigma\mu} Z^{\cdot\mu}_{\sigma\cdot} q^{\sigma}_{\cdot\mu} .$$
$$+ (p-1) \sum_{\mu\sigma} Z^{\cdot\mu}_{\sigma\cdot} q_{\mu} q^{\sigma}_{\cdot\lambda} , \qquad (1.5)$$

and

$$(q^2 - 2pq + 2p) \Big[\sum_{\mu} \nabla_{\mu} Y^{\mu}_{\cdot\lambda} - (q-1) \sum_{\sigma\mu} Y^{\mu\cdot}_{\cdot\sigma} q^{\sigma} q_{\lambda\mu}$$
$$(q-1) q^{\lambda} \sum_{\sigma\mu} Y^{\mu\cdot}_{\cdot\sigma} q^{\sigma}_{\cdot\mu} + (p-1) \sum_{\sigma\mu} Y^{\mu}_{\cdot\sigma} q_{\mu} q^{\sigma}_{\cdot\lambda} \Big] \qquad (1.6)$$

Making use of the following relations, (which we prove in the following sections)

$$\sum_{\mu} \nabla_{\mu} (N^{\cdot\mu}_{\lambda\cdot} - \tfrac{1}{2} N \delta^{\mu}_{\lambda})$$
$$= (3pq - q^2 - 2p) \sum_{\rho\sigma} q_{\lambda\sigma} \nabla_{\rho} q^{\rho\sigma} + (q^2 - 2pq + 2p)$$
$$\sum q^{\sigma\mu} \nabla_{\lambda} q_{\mu\sigma} + q(p-q) \sum q^{\sigma\rho} \nabla_{\rho} q_{\sigma\lambda} , \qquad (1.7)$$

$$\sum_{\mu\rho} \nabla_{\mu} \overset{R}{\nabla}_{\rho} q^{\rho\mu} = 0, \qquad (1.8)$$

$$\sum_{\sigma\mu} q^{\sigma\mu} \nabla_{\lambda} q_{\sigma\mu} = 2 \sum_{\rho\sigma} q^{\rho\sigma} \nabla_{\rho} q_{\lambda\sigma} , \qquad (1.9)$$

in evaluating (1.5), we see that it reduces to

$$2(q^2 - 2pq + 2p)\, q_{\lambda\mu} \sum \nabla_{\rho} q^{\mu\rho}$$

Similarly, after reduction, (1.6) comes out to be equal to

$$2(q^2 - 2pq + 2p) \sum_{\mu\rho} q_{\mu\lambda} \nabla_\rho q^{\mu\rho}.$$

So the left side of the identities reduces to

$$2(q^2 - 2pq + 2p) q_{\lambda\mu} \sum_\rho \nabla_\rho q^{\mu\rho} + 2(q^2 - 2pq + 2p) q_{\lambda\mu} \sum_\rho \nabla_\rho q^{\mu\rho},$$

and this is zero since $q_{\mu\lambda} = - q_{\lambda\mu}$. Thus the identities are verified to be true by actual substitution.

2. To prove (1.7) we make use of the following relation given by Schouten and van Dantzig.*

$$\sum_\mu \nabla_\mu [N_{\lambda\cdot}^{\cdot\mu} - \tfrac{1}{2} N \delta_\lambda^\mu] = -2 \sum_{\mu\sigma} S_{\lambda\cdot\sigma}^{\cdot\mu} N_{\mu\cdot}^{\cdot\sigma} - \sum_{\mu\rho\sigma} S_{\rho\sigma\mu} N_\lambda^{\cdot\mu\rho\sigma}$$

$$= -2 \sum_{\mu\sigma} \left[-(q-1) q_{\lambda\sigma} q^\mu + \frac{p-q}{2} q_\lambda q_\sigma^{\cdot\mu} - \frac{p-q}{2} q_\sigma q_\lambda^{\cdot\mu} \right] N_\mu^{\cdot\sigma}$$

$$- \sum_{\mu\rho\sigma} \left[-(q-1) q_{\rho\sigma} q_\mu + \frac{p-q}{2} q_\rho q_{\sigma\mu} - \frac{p-q}{2} q_\sigma q_{\rho\mu} \right] N_\lambda^{\cdot\mu\rho\sigma}, \qquad (2.1)$$

remembering the values of $S_{\lambda\cdot\sigma}^{\cdot\mu}$, given in the first lecture.

* *Annals of Mathematics*, 34 (1933), 293.

The first term on the right of (2.1) contains transvections of the q's with $N_{\mu\cdot}^{\cdot\sigma}$. To evaluate these we make use of the following identities, which can be easily obtained from those given by Schouten and van Dantzig in their paper in the *Annals of Mathematics* :

$$\begin{aligned}
&\sum_{\rho} q^{\rho} N_{\rho\sigma} = p \sum_{\rho} \nabla_{\rho} q^{\rho}_{\cdot\sigma}, \\
&\sum_{\sigma} q^{\sigma} N_{\rho\sigma} = q \sum_{\tau} \nabla_{\tau} q^{\tau}_{\cdot\rho} + q(p-q) q_{\rho} \sum_{\theta\tau} q^{\theta\tau} q_{\theta\tau}, \\
&\sum_{\sigma\rho} q^{\sigma\rho} N_{\rho\sigma} = 0. \qquad (2.2)
\end{aligned}$$

To evaluate the second term of (2.1) we observe that

$$\begin{aligned}
N_{\lambda}^{\cdot\mu\rho\sigma} \cdot \quad K_{\lambda}^{\cdot\mu\rho\sigma} = {} & qq^{\rho\mu\theta} \overset{R}{\nabla}_{\theta} q^{\sigma}_{\cdot\lambda} - qq^{\sigma} G^{\mu\theta} \overset{R}{\nabla}_{\theta} q^{\rho}_{\cdot\lambda} \\
& + pq_{\lambda} G^{\mu\theta} \overset{R}{\nabla}_{\theta} q^{\sigma\rho} \\
& - qq^{\rho} \overset{R}{\nabla}_{\lambda} q^{\sigma\mu} + qq^{\sigma} \nabla_{\lambda} q^{\rho\mu} - pq^{\mu} \nabla_{\lambda} q^{\sigma\rho} \\
& + q^{2} q^{\sigma\mu} q^{\rho}_{\cdot\lambda} - 2pq^{\mu}_{\lambda\cdot} q^{\sigma\rho} \\
& + pq[-q^{\sigma} q_{\lambda} q^{\alpha\rho} q^{\mu}_{\alpha\cdot} + q^{\rho} q^{\mu} q^{\sigma\alpha} q_{\alpha\lambda} + q^{\sigma} q^{\mu} q_{\alpha\lambda} q^{\rho\alpha} \\
& - q^{\rho} q_{\lambda} q_{\cdot\alpha} q^{\alpha\mu}]. \qquad (2.3)
\end{aligned}$$

Transvection of the $K_{\lambda}^{\cdot\mu\rho\sigma}$ in (2.3) with the q's vanishes since $K_{\lambda\cdots}^{\cdot\mu\rho\sigma}$ is an affinor, when we substitute in the second term of (2.1) from (2.3). The transvections of the right side of (2.3) with the q's can be easily evaluated by the known formulæ

in the first two lectures. After these simplifications, (2.1) comes out as

$$\sum_{\mu} \nabla_\mu (N^{\mu}_{\lambda :} - \tfrac{1}{2} N \delta^{\mu}_{\lambda}) = (3pq - q^2 - 2p) \sum_{\rho\sigma} q_{\lambda\sigma} \nabla_\rho q^{\rho\sigma}$$
$$+ (q^2 - 2pq + 2p) \sum_{\mu\sigma} q^{\sigma\mu} \nabla_\lambda q_{\sigma\mu} + q(p-q) \sum_{\rho\sigma} q^{\sigma\rho} \nabla_\rho q_{\sigma\lambda},$$

which is (1.7).

3. *To prove that* $\sum_{\mu\rho} \nabla_\mu \overset{\mu\nu}{\nabla}_\rho q^{\rho\mu} = 0$

We have

$$\sum_{\rho} \overset{\mu\nu}{\nabla}_\rho q^{\mu\rho} = \sum_{\rho} \nabla_\rho q^{\mu\rho} + q q^{\mu} \sum_{\rho\sigma} q^{\rho\sigma} q_{\rho\sigma}. \qquad (3.1)$$

This comes out by remembering that

$$\overset{R}{\nabla}_\mu q^{\nu}_{.\lambda} = \sum_{\pi\rho\sigma} (\delta^{\pi}_{\lambda} + q^{\pi} q_{\lambda})(\delta^{\rho}_{\mu} + q^{\rho} q_{\mu})(\delta^{\nu}_{\sigma} + q^{\nu} q_{\sigma}) \nabla_\rho q^{\sigma}_{.\lambda}$$
$$= \nabla_\mu q^{\nu}_{.\lambda} + q q^{\nu} \sum_{\rho} q_{\rho\lambda} q^{\rho}_{.\mu} + q q_{\lambda} \sum_{\rho} q^{\nu\rho} q_{\rho\sigma} \qquad (3.2)$$

which follows after some simplification.

In getting (3.2) we require a formula

$$\sum_{\rho} q^{\rho} \nabla_\rho q^{\nu}_{.\lambda} = 0, \qquad (3.3)$$

which can be easily seen to be true because

$$\sum_{\rho} q^{\rho} \nabla_\rho q^{\nu}_{.\lambda} = \omega^{-1} \sum_{\rho} x^{\rho} \frac{\partial q^{\nu}_{.\lambda}}{\partial x^{\rho}} - \sum_{\rho\sigma} q^{\rho} \Pi^{\sigma}_{\lambda\rho} q^{\nu}_{.\sigma} + \sum_{\rho\sigma} q^{\rho} \Pi^{\nu}_{\sigma\rho} q^{\sigma}_{.\lambda}.$$

The first term on the right vanishes and

$$\sum_{\rho} \Pi^{\sigma}_{\lambda\rho} q^{\rho} = \omega^{-1} (P^{\sigma}_{.\lambda} - \delta^{\sigma}_{\lambda})$$
$$\sum \Pi^{\nu}_{\sigma\rho} q^{\rho} = \omega^{-1} (P^{\nu}_{.\sigma} - \delta^{\nu}_{\sigma})$$

give

$$\sum_{\rho} q^{\rho} \nabla_{\rho} q^{\nu}_{.\lambda} = - \sum_{\rho} \omega^{-1}(P^{\sigma}_{.\lambda} - \delta^{\sigma}_{\lambda}) q^{\nu}_{.\sigma} + \sum_{\sigma} \omega^{-1}(P^{\nu}_{.\sigma} - \delta^{\nu}_{\sigma}) q^{\sigma}_{.\lambda} = 0.$$

We also require for (3.2)

$$\sum_{\rho} q^{\rho} \nabla_{\mu} q^{\nu}_{.\rho} = q \sum_{\rho} q^{\nu}_{\rho}.q^{\rho}_{.\mu} \qquad \ldots\ldots\ldots \quad (3.4)$$

$$\sum_{\sigma} q_{\sigma} \nabla_{\mu} q^{\sigma}_{.\lambda} = q \sum_{\rho} q_{\rho\lambda} q^{\rho}_{.\mu} \qquad \ldots\ldots\ldots \quad (3.5)$$

both of which follow from

$$\sum_{\rho} q^{\rho} q^{\nu}_{.\rho} = 0.$$

We get (3.1) from (3.2) by raising λ, changing it into $\lambda = \mu = \rho$ and putting μ for ν.

Operating on both sides of (5.1) by ∇_{μ} we have

$$\sum_{\mu\rho} \nabla_{\mu} \overset{R}{\nabla}_{\rho} q^{\mu\rho} = \sum_{\mu\rho} \nabla_{\mu} \nabla_{\rho} q^{\mu\rho} + \sum_{\mu} q q^{\mu} \nabla_{\mu} \{ \sum_{\rho\sigma} q^{\rho\sigma} q_{\rho\sigma} \}. \qquad \ldots\ldots\ldots\ldots \quad (3.6)$$

Both the terms on the right side of (3.6) can be seen to be equal to zero, as by means of (3.3)

$$\sum q^{\mu} \nabla_{\mu} \sum_{\rho\sigma} q^{\rho\sigma} q_{\rho\sigma} = 0;$$

$$\sum_{\mu\rho} \nabla_{\mu} \nabla_{\rho} q^{\mu\rho} = \sum_{\mu\rho\tau\beta} \nabla_{\mu} \nabla_{\rho} [\tfrac{1}{2} G^{\theta\mu} G^{\tau\rho} (\nabla_{\theta} q_{\tau} - \nabla_{\tau} q_{\theta})]$$

$$= \sum_{\mu\rho\theta} \tfrac{1}{2} G^{\mu\theta} \nabla_{\mu} \nabla_{\rho} \nabla_{\theta} q^{\rho} - \tfrac{1}{2} \sum_{\mu\rho\tau} G^{\tau\rho} \nabla_{\rho} \nabla_{\mu} \nabla_{\tau} . q^{\mu} = 0$$

by interchanging μ and ρ and changing the dummy suffix θ into τ in the first term on the right.

4. *To prove that* $\sum\limits_{\sigma\mu} g^{\sigma\mu} \nabla_\lambda g_{\sigma\mu} = 2\sum\limits_{\rho\sigma} g^{\rho\sigma} \nabla_\rho g_{\lambda\sigma}$.

$$\sum_{\sigma\mu} g^{\sigma\mu} \nabla_\lambda g_{\sigma\mu} = \sum_{\sigma\mu} g^{\sigma\mu} \left[\frac{\partial g_{\sigma\mu}}{\partial x^\lambda} - \sum_\rho \Pi^\rho_{\sigma\lambda} g_{\rho\mu} - \sum_\rho \Pi^\rho_{\mu\lambda} g_{\sigma\rho} \right]$$

$$= \sum_{\sigma\mu} g^{\sigma\mu} \frac{\partial g_{\sigma\mu}}{\partial x^\lambda} - 2\sum_{\mu\rho\sigma} g_{\rho\sigma} g^{\mu\sigma} \begin{Bmatrix} \rho \\ \lambda\mu \end{Bmatrix}, \quad (4.1)$$

$$2\sum_{\rho\sigma} g^{\rho\sigma} \nabla_\rho g_{\lambda\sigma} = 2\sum_{\rho\sigma} g^{\rho\sigma} \left[\frac{\partial g_{\lambda\sigma}}{\partial x^\rho} - \sum_\tau \Pi^\tau_{\lambda\rho} g_{\tau\sigma} - \sum_\tau \Pi^\tau_{\sigma\rho} g_{\lambda\tau} \right]$$

$$= 2\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial g_{\lambda\sigma}}{\partial x^\rho} - 2\sum_{\tau\rho\sigma} g_{\tau\sigma} g^{\rho\sigma} \begin{Bmatrix} \tau \\ \lambda\rho \end{Bmatrix}$$

$$= 2\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial g_{\lambda\sigma}}{\partial x^\rho} - 2\sum_{\mu\rho\sigma} g_{\sigma\rho} g^{\mu\sigma} \begin{Bmatrix} \rho \\ \mu\lambda \end{Bmatrix} \quad \ldots\ldots\ldots \quad (4.2)$$

Now
$$2\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial g_{\lambda\sigma}}{\partial x^\rho} = \sum_{\sigma\rho} g^{\rho\sigma} \frac{\partial}{\partial x^\rho} \left\{ \frac{\partial g_\sigma}{\partial x^\lambda} - \frac{\partial g_\lambda}{\partial x^\sigma} \right\}$$

$$= \sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial^2 g_\sigma}{\partial x^\rho\, \partial x^\lambda}, \quad \ldots\ldots \quad (4.3)$$

$$\sum_{\sigma\mu} g^{\sigma\mu} \frac{\partial g^{\sigma\mu}}{\partial x^\lambda} = \tfrac{1}{2}\sum_{\sigma\mu} g^{\sigma\mu} \frac{\partial^2 g_\mu}{\partial x^\lambda\, \partial x^\sigma} - \tfrac{1}{2}\sum_{\sigma\mu} g^{\sigma\mu} \frac{\partial^2 g_\sigma}{\partial x^\lambda\, \partial x^\mu}$$

$$= \tfrac{1}{2}\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial^2 g_\sigma}{\partial x^\lambda\, \partial x^\rho} + \tfrac{1}{2}\sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial^2 g_\sigma}{\partial x^\lambda\, \partial x^\rho}$$

$$= \sum_{\rho\sigma} g^{\rho\sigma} \frac{\partial^2 g_\sigma}{\partial x^\rho\, \partial x^\lambda}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.4)$$

Hence $$\sum_{\sigma\mu} q^{\sigma\mu} \nabla_\lambda q_{\sigma\mu} = 2 \sum_{\rho\sigma} q^{\rho\sigma} \nabla_\rho q^{\lambda\sigma}$$

by means of (4.4) and (4.3).

5. *Connection with identities between field-equations in Einstein's general relativity.*

True affinors give the tensors in four-dimensions. Hence finding the true affinor corresponding to the left side of the identities and putting it equal to zero we have

$$\sum_j \overset{R}{\nabla}_j Z_i^j + 2(q^2 - 2pq + 2p) \sum_{jl} q_{ji} \overset{R}{\nabla}_l q^{jl} = 0$$

$$(j, i, l = 1, 2, 3, 4). \quad \ldots\ldots \quad (5.1)$$

This must give us the identities between field-equations in Einstein's general theory of relativity.

It is easy to see that

$$Z_{i.}^{\cdot j} = K_{i.}^{\cdot j} - \tfrac{1}{2} K \delta_i^j + \frac{K}{c^2}\Big(\sum_{k=1} F_{i.}^{\cdot k} F_{\cdot k}^{j.} - \tfrac{1}{4}\delta_i^j \sum_{k,l=1}^{4} F_{kl} F^{kl}\Big)$$

$$= A_{i.}^{\cdot j} \text{ say } (i, j = 1, 2, 3, 4). \quad \ldots\ldots\ldots\ldots\ldots \quad (5.2)$$

Then (5.1) becomes

$$\sum_{j=1}^{4} \overset{R}{\nabla}_j A_i^j + \frac{K}{c^2} \sum_{j=1}^{4} F_{ij} B^j = 0 \; (i = 1, 2, 3, 4);$$

or $$\sum (A_i^j)_j + \frac{K}{c^2} \sum_{j=1}^{4} F_{ij} B^j = 0; \; (i = 1,2,3,4), \quad (5.3)$$

where $$B^j = \sum_i \overset{R}{\nabla}_i F^{ji} = 0, \; (i, j = 1, 2, 3, 4)$$

gives Maxwell's equations and $A_i^i = 0$ gives Einstein's combined gravitational and electromagnetic field-equations.

6. E. T. Whittaker at the end of his paper* on "Hilbert's world-function" gives the identities between field-equations of Einstein's general theory of relativity. When we assume that there are no currents and no massive particles his identities reduce to our identities (5.3) above.

In vacuo, the identities given by Whittaker are

$$A_p = \sum_q B^q M_{pq},\ p,\ q = 0, 1, 2, 3 \ \ldots\ldots\ldots \quad (6.1)$$

where A_p represents the vectorial divergence of the symmetrical tensor A_{pq} which is equal to

$$\gamma(K_{pq} - \tfrac{1}{2}g_{pq}K) + \tfrac{1}{2}\left[\tfrac{1}{4}g_{pq}\sum_{r,s} X_{rs}X^{rs} - \sum_s X_{qs}X^{:s}_{.p}\right]$$

$$X_{rs} = M_{rs} = \frac{\partial\phi_r}{\partial x^s} - \frac{\partial\phi_s}{\partial x^r} = F_{rs} \text{ above}$$

$$B^q = \frac{1}{\sqrt{-g}}\sum_q \frac{\partial X^{pq}}{\partial X^q},$$

so that $B^q = 0$ gives Maxwell's equations, and γ is a constant inversely proportional to the Newtonian constant of gravitation. Taking $\gamma = -\tfrac{1}{2}c^2/\kappa$ (6.1) and (5.3) can be at once seen to be the same.

Proc. Royal Society, London, (A), 113 (1927), 496.

7. J. M. Whittaker* at the end of his paper gives such identities in general relativity taking into account the wave-mechanical considerations. His identities, in space free from electric and magnetic currents and matter and also free from wave-mechanical restrictions become

$$\sum_{\nu} (A^{\nu}_{\mu})_{\nu} - \sum_{\nu} B^{\nu} X_{\nu\,\mu} = 0 (\mu, \nu = 1, 2, 3, 4), \quad \ldots \quad (7.1)$$

where

$$\tfrac{1}{2} A^{\mu\,\nu} = \gamma(K^{\mu\,\nu} - \tfrac{1}{2} K g^{\mu\,\nu}) + \tfrac{1}{2} E^{\mu\,\nu},$$

$$B^{\mu} = (X^{\mu\,\nu})_{\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\nu}} (X^{\mu\,\nu} \sqrt{-g})$$

and
$$X_{\mu\,\nu} = \frac{\partial \phi_{\nu}}{\partial x^{\mu}} - \frac{\partial \phi_{\mu}}{\partial x^{\nu}},$$

ϕ_{μ} being the electromagnetic potential.

Taking $\gamma = -\frac{1}{2} c^2/\kappa$, (7.1) can at once be seen to be the same as (5.3) above.

8. W. Pauli† in his unified field-theory gives the following identities between field equations :

$$\sum_{k} K^{,k}_{i;k} - \varepsilon \sum_{k} X_{ik} K^{k}_{\cdot(0)} \equiv 0 \ (k = 1, 2, 3, 4), \quad \ldots \quad (8.1)$$

* 'On the Principle of least action,' *Proc. Royal Society of London* (A), 121 (1928), 543-57.

† 'Über die Formulierung der Naturgesetze mit fünf homogenen Koordinaten,' *Ann. der Ph.*, 18 (1933), 305-72.

where

$$K_{ij} = R_{ij} - \tfrac{1}{2}g_{ij} + \frac{\varkappa}{c^2}\sum_r \left(F_{i\cdot}^{\cdot r}F_{jr} - \tfrac{1}{4}g_{ij}\sum_{rs} F^{rs}F_{rs}\right)$$

$$K^{i}_{\cdot(0)} = -\frac{1}{r}\frac{\sqrt{\varkappa}}{c}\sum_k F^{ik}_{:k}$$

$$X^{ik} = r\frac{\sqrt{\varkappa}}{c} F_{ik}$$

and $\varepsilon = \pm 1$.

Pauli's K_{ij} is A_{ij} of (5.2) and his R_{ij} is our K_{ij}. Thus (8.1) in our notation becomes

$$\sum_k (A^k_i)_k + \frac{\varkappa}{c^2}\sum_{kj} F_{ik}F^{kj}_{\cdot j} = 0$$

or

$$\sum_j (A^j_i)_j + \frac{\varkappa}{c^2}\sum_j F_{ij}B^j = 0, \qquad \dots \qquad (8.2)$$

where $B^j = \sum_k F^{jk}_{k} = \sum_k \overset{R}{\nabla}_k F^{jk} = \sum_i \overset{R}{\nabla}_i F^{ji}$.

Thus (8.2) is the same as (5.3).

9. *Introduction of* Λ *into the identities*—We shall now try to introduce, Λ, the universal constant proportional to the square of curvature of the world, into the field-equations and identities given in the second lecture.

If in the variational integral for obtaining the field equations we take $N - 2\Lambda$ for N and proceed

in the same manner we arrive at

$$K_{\lambda\mu} - \tfrac{1}{2}Kg_{\lambda\mu} + \Lambda g_{\lambda\mu} + (q^2 + 2pq + 2p)$$
$$\times \{\tfrac{1}{2}g_{\lambda\mu}\sum_{\rho\sigma} q_{\rho\sigma}q^{\rho\sigma} - 2\sum_{\rho} q^{\cdot\rho}_{\lambda}\, q_{\mu\rho}\} = 0 \quad (\lambda,\mu = 0,1,2,3,4)$$
$$\ldots\ldots\ldots\ldots \quad (9.1)$$

$X_{\lambda\mu}$ of the identities now takes the form

$$\overline{X}_{\lambda\mu} = Z_{\lambda\mu} + \Lambda g_{\lambda\mu} + (q^2 - 2pq + 2p)\, Y_{\lambda\mu} \quad (9.2)$$

The identities then take the form

$$\sum_{\lambda\mu}[\nabla_{\mu}\overline{X}^{\mu}_{\cdot\alpha} - \{(q-1)\, q^{\lambda} q_{\alpha\mu} - (q-1) q_{\alpha} q^{\lambda}_{\cdot\mu}$$
$$- (p-1)\, q_{\mu} q^{\lambda}_{\cdot\alpha}\}\overline{X}^{\mu}_{\lambda}] = 0 \; (\alpha = 0, 1,2,3,4) \quad (9.3)$$

where $\overline{X}_{\lambda\mu}$ is given by (9.2) above.

Finding the true affinor corresponding to the left side of (9.3) we have

where

$$\sum_{j} \overset{R}{\nabla}_{j}\, \overline{Z}^{j}_{i} + (2q^2 - 2pq + 2p) \sum_{jl} q_{ji} \overset{R}{\nabla}_{l}\, q^{jl} = 0$$
$$(j,i = 1, 2, 3, 4), \ldots\ldots\ldots \quad (9.4)$$

$$\overline{Z}_{ij} = K_{ij} - \tfrac{1}{2}K g_{ij} + \Lambda g_{ij} + (q^2 - 2pq - 2p)$$
$$\{\tfrac{1}{2}\, g_{ij}\sum_{k,l=1}^{4} q_{kl}\, q^{kl} - 2\sum_{l=1}^{4} q^{l}_{i} q_{jl}\}$$
$$= K_{ij} - \tfrac{1}{2}Kg_{ij} + \Lambda g_{ij} + \frac{\varkappa}{c^2}\Big(\sum_{k=1}^{4} F^{\cdot k}_{i\cdot}F_{jk}$$
$$- \tfrac{1}{4}\, g_{ij}\sum_{k,l=1}^{4} F_{kl}\, F^{kl}\Big)$$
$$= A_{ij} \text{ say.}$$

(9.4) can be written as

$$\sum_j \overset{R}{\nabla}_j A^j_i + \frac{\varkappa}{c^2} \sum_j F_{ij} B^j = 0$$

or

$$\sum_j (A^j_i)_j + \frac{\varkappa}{c^2} \sum_j F_{ij} B^j = 0 \quad \ldots\ldots\ldots \quad (7.5)$$

where $A_{ij} = 0$ gives Einstein's combined electromagnetic field-equations with Λ introduced into them, and $B^j = \sum_i \overset{R}{\nabla}_j F^{ji}$ $(i, j = 1, 2, 3, 4)$ gives Maxwell's Equations.

10. *Introduction of the current vector*—The term with the current vector in the Maxwell's equations has been introduced in the second lecture, taking the case $p = 2q$. The relation (9.5) can be written in this case (when $p = 2q$) as

$$(A^j_i)_j + \frac{\varkappa}{c^2} F_{ij} \bar{B}^j = 0 \quad \ldots\ldots \quad (10.1)$$

where

$$\bar{B}^j = \overset{R}{\nabla}_i F^{ij} - es^j = 0 \quad \ldots\ldots \quad (10.2)$$

gives Maxwell's equation with the current vector es_j introduced. It is easy to see that (9.5) can be put into the form (10.1) because

$$\sum_j F_{ij} s^j = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (10.3)$$

which may be seen to be true by converting it into homogeneous coordinates.

In homogeneous coordinates the left-side of (10.3) becomes

$$-\sum_{\mu} F_{\mu\lambda} \cdot e \cdot s \cdot q^{\mu} \quad \ldots\ldots\ldots\ldots \quad (10.4)$$

or

$$-\frac{qesc}{k}\sum_{\mu} g_{\lambda\mu} q^{\mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10.5)$$

which is zero.

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